PhD Macro
Professor Anne Sibert
Due: 2 February

Question 1. max $\int_0^1 (x+y)\,dt$ subject to $\dot{x} = 1 - y^2$ and $x(0) = 1$.

Question 2. min $\int_0^1 y^2 dt$ subject to $\dot{x} = x + y$ and $x(0) = 1$.

Question 3. Consider a closed economy representative agent model with production.

Time goes to infinity and there is perfect foresight. Competitive firms convert capital and labour into output via a neo-classical constant-returns-to-scale production function. There is no depreciation.

$$Y = Lf(k), \tag{1}$$

where L is labour and k is the capital-labour ratio. We assume that there is strictly positive and strictly diminishing marginal productivity of capital and that the Inada conditions are satisfied. Competitiveness then implies

$$\begin{aligned} f'(k) &= r && (2)\\ f(k) - kf'(k) &= w, && (3) \end{aligned}$$

where r is the interest rate and w is the wage rate.
(A) Demonstrate that solving (2) and (3) yields

$$w = w(k)\,,\ r = r(k)\,, \text{ where}$$

$$\begin{aligned} \frac{kr'(k)}{r(k)} &= -\sigma < 0, \text{ where } \sigma := -\frac{kf''(k)}{f'(k)} \\ \frac{kw'(k)}{w(k)} &= \frac{\sigma(1-\theta)}{\theta} > 0, \text{ where } \theta := \frac{wL}{F} = \text{labour's share of output.} \end{aligned}$$

(B) The representative household solves the problem

$$\max \int_0^\infty e^{-\delta t}\left[\alpha \ln c + (1-\alpha)\ln\left(\bar{L} - L\right)\right] dt,\ \delta > 0,$$
$$\text{subject to } \dot{a} = ra + wL - T - c.$$

where c is consumption, s is savings of bonds and capital, T is taxes and a_0 is given. Find the Euler equations for the household.

Assume that the terminal condition is

$$\lim_{t\to\infty} \lambda a^{-\delta t} = 0. \tag{4}$$

(C) The government finances its deficits by issuing real bonds

$$\dot{b} = g + rb - T. \tag{5}$$

Putting this together with the private budget constraint yields the economy-wide feasibility or market-clearing condition:

$$\dot{k} = Lf\left(k\right) - c - g. \tag{6}$$

Using (6) and your results for the household, express the equilibrium as two first-order difference equations in λ and k.
(E) Linearise around the steady state and analyse the equilibrium algebraically.
(F) Draw the phase diagram and use it to analyse the equilibrium.
(G) Uses the phase diagram to analyse what happens if the economy is at a steady state and there is an (unexpected) increase in δ.