Lecture 1: The Classical Optimal Growth Model

This lecture introduces the classical optimal economic growth problem. Solving the problem will require a dynamic optimisation technique: a simple calculus of variations procedure will do. I describe how the first-order conditions are found and discuss the second-order conditions. The outcome of the optimisation will be a pair of non-linear second-order ordinary differential equations. I will consider a linear approximation around the steady state. I describe how to solve a pair of linear second-order differential equations and how to analyse them graphically with a phase diagram. Finally, I consider some comparative dynamics.

1 Setting up the problem

A closed economy produces a single capital/consumption good. Firms transform capital and labour into output via a neoclassical production function. Time is continuous and goes to infinity.

1.1 The neoclassical production function

Three equations summarise the main features of the economy. First, output is a function of capital and labour:

$$Y = F(K, L), \tag{1}$$

where Y is output, K is the capital stock and L is labour. Variables are functions of time but this is omitted for notational convenience. Second, output can be used for consumption or for investment to increase the capital stock:

$$Y = C + I, \tag{2}$$

where C is consumption and I is investment. Third, the change in the capital stock is equal to investment minus depreciation, which occurs at the constant rate $u > 0$,

$$\dot{K} = I - uK. \tag{3}$$

It is assumed that the production function exhibits constant returns to scale. Recall that this means that for every constant $\alpha > 0$, $F(\alpha K, \alpha L) = \alpha F(K, L)$. In particular, if $\alpha = 1/L$, $F(K/L, 1) = F(K, L)/L$. This means that output per unit of labour can be written as a function of the capital-labour ratio.

I use the notational conventional that variables and functions denoted by lower case letters are the ratio of the corresponding variables and functions denoted by upper case letters to labour so that, for example, $y := Y/L$. Thus, $y = f(k) := F(k, 1)$. Note that $\dot{k} = d(K/L)/dt = \dot{K}/L - (K/L)\dot{L}/L = \dot{K}/L - k\dot{L}/L$.

Assume that labour grows at a constant rate $n > 0$. Thus $\dot{L}/L = n$ and $\dot{K}/L = \dot{k} + kn$.

Equations (1) - (3) can be rewritten as

$$y = f(k) \tag{4}$$

$$y = c + i \tag{5}$$

$$\dot{k} = i - (u + n) k. \tag{6}$$

1.2 Properties of the equilibrium when c is exogenous

This subsection considers the properties of the model when consumption is exogenous.

By equations (4) - (6),

$$\dot{k} = f(k) - (u + n) k - c. \tag{7}$$

A stationary solution to equation (7) has $\dot{k} = 0$, and hence, $c = f(k) - (u + n) k$. This is shown in Figure 1, below.

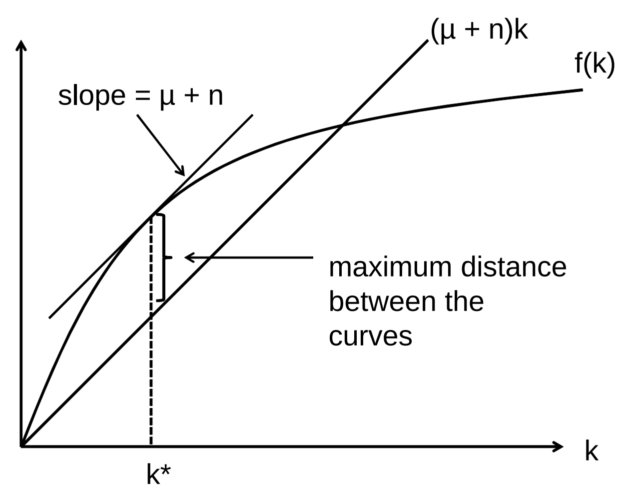

Figure 1. Steady State

In Figure 1, consumption is equal to the distance between the bowed curve $f(k)$ and the line $(u + n) k$. Some additional assumptions are needed to ensure that things look as they do in the graph. It is assumed that $f(k)$ is twice differentiable on the strictly positive part of the real line $\mathbb{R}_{++}$; that it is strictly increasing (i.e., that there is a strictly positive marginal productivity of capital) and strictly concave (i.e., $f''(k) < 0$ and there is strictly decreasing marginal productivity of capital). It is assumed that output is zero when capital is zero: $f(0) = 0$.

To ensure that there is some interval of capital-labour ratios such that $f(k) > (u+n)k$, it is sufficient to impose the Inada condition that $f'(k) \to \infty$ as $k \to 0$. I also want some $\hat{k}$ such that $f\left(\hat{k}\right) = (u+n)\hat{k}$ That is, I want to avoid the following, where $f'(k) \to u+n$.

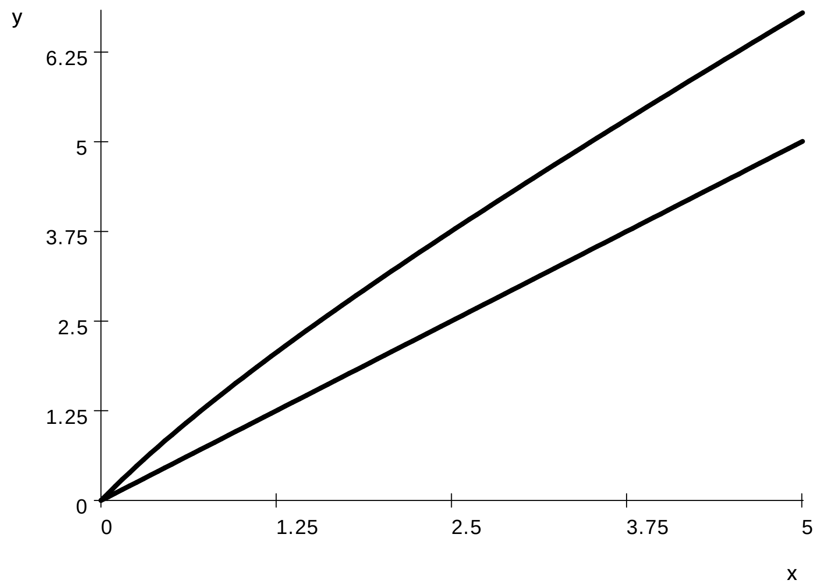

Figure 2. Inada Condition Needed

This can be done by imposing the Inada condition that $f'(k) \to 0$ as $k \to \infty$.

It is seen from Figure 1 that $f(k) - (u+n)k$ is maximised at the unique k^* such that $f'(k^*) = u+n$. This capital-labour ratio is know as the *golden rule* capital-labour ratio and it permits the largest sustainable level of consumption, $c^* := f(k^*) - (\mu+n)k^*$. The idea of the golden rule is due to Phelps (1961).[1]

Suppose that $c = c^*$. Then the capital-labour ratio k^* that solves the first-order differential equation (7) is not stable. This is seen in Figure 3 below.

[1] Solow (1956) is the classical model analysing growth in this one-sector model. He assumed that $\dot{K} = sY$.

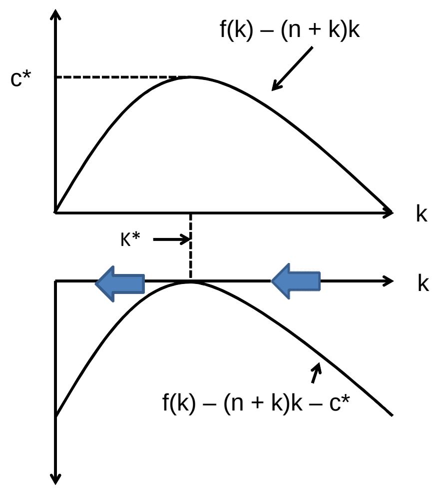

Figure 3. Unstable Outcome

In the upper part of Figure 3, $f(k) - (u+n)k$ equals zero at zero, increases to c^* at $k = k^*$ and is strictly decreasing after this point. The curve $\dot{k} = f(k) - (u+n)k - c^*$ is depicted as a function of k in the bottom part of Figure 3. This curve is strictly negative except at $k = k^*$, where it equals zero. Since $\dot{k} < 0$ except at $k = k^*$, k is decreasing everywhere except at $k = k^*$.Thus there is an unstable stationary point at k^*.

Note that Figure 3 is an example of how you can do a graphical analysis of a first-order autonomous differential equation $dy/dx = f(y)$. Graph $f(y)$ and this tells you where y is increasing and where it is decreasing.

1.3 The social planner's problem

Suppose that the social planner's problem is

$$\max \int_0^\infty e^{-\delta t} U(c)\, dt,\ 0 < \delta \tag{8}$$

subject to

$$\dot{k} = f(k) - (u+n)k - c, \tag{9}$$

$k \geq 0$, $c \geq 0$ and k_0 given.

The discount rate δ ensures that more weight is put on the present population. It may also reflect population growth. If the discount rate applied to an individual worker is ξ and the population grows at rate n then $\delta = \xi - n$.

The specification of an objective function as in problem (8) seems natural to economists now but earlier economists did not view this as an obvious choice.[2] Ramsey (1928), for example, thought that a moral planner should weight all generations equally. He then posited that people had a bliss point. So the optimisation problem was to get all generations to the bliss point as quickly as possible. Cass (1965) is an early example of a paper where the objective function is specified as it is here.

The problem under consideration is of a type referred to as a *control problem*. One approach to solving control problems is that of the calculus of variations. The next section considers how to use this technique to solve our problem.

2 Calculus of Variations

The calculus of variations is a 17$^{\text{th}}$ century approach to solving optimisation problems. It originated with Johann Bernoulli's study of the brachistochrone curve problem in 1696.[3] Early contributors include Jakob Bernoulli, Guillaume François Antoine, Marquis de l'Hôpital and Leonhard Euler, whose 1766 paper *Elementa Calculi Variationum* supplied the name.

Consider the problem:

$$\begin{aligned} &\max_x \int_{t_0}^{t_1} I\left(x\left(t\right), \dot{x}\left(t\right), t\right) dt \qquad (10) \\ &\text{subject to } x\left(t_0\right) = x_0,\ x\left(t_1\right) = x_1, \end{aligned}$$

where the function I is assumed to be continuous and to have continuous partial derivatives with respect to its first two arguments. It is also required that a solution x be continuously differentiable on $[x_0, x_1]$.[4]

2.1 The Euler Equation

Our first step here is to find the analogue to the static first-order conditions for this dynamic problem in (10). To do this, suppose that $x^*(t)$ is a solution to the maximisation problem. Let $z(t) = x^*(t) + \epsilon v(t)$, where $v(t)$ is continuously differentiable on $[x_0, x_1]$ and $v(t_0) = v(t_1) = 0$. Then the function z is continuously

[2] See Koopmans (1965) for a discussion of this.

[3] The brachistochrone problem is to find the path, or curve, that will carry a bead resting at one point to another in the least amount of time, assuming that the bead moves under the force of gravity and without friction. See http://mathworld.wolfram.com/BrachistochroneProblem.html.

[4] Good references are Intriligator (1971) and especially the wonderfully clear Kamien and Schwartz (1981).

differentiable on $[x_0, x_1]$ and satisfies the boundary conditions. Let

$$J(\epsilon) = \int_{t_0}^{t_1} I(z(t), \dot{z}(t), t)\, dt = \int_{t_0}^{t_1} I(x^*(t) + \epsilon v(t), \dot{x}^*(t) + \epsilon \dot{v}(t), t)\, dt. \tag{11}$$

Since $x^*(t)$ solves the maximisation problem (10), J must be maximised at $\epsilon = 0$. The first-order condition for this static optimisation problem is $J'(\epsilon) = 0$. Using the chain rule yields

$$0 = J'(\epsilon) = \int_{t_0}^{t_1} \frac{\partial I}{\partial x} v dt + \int_{t_0}^{t_1} \frac{\partial I}{\partial \dot{x}} \dot{v} dt. \tag{12}$$

Integrating by parts to evaluate the second integral yields[5]

$$0 = J'(0) = \int_{t_0}^{t_1} \frac{\partial I}{\partial x} v dt + \left. \frac{\partial I}{\partial \dot{x}} v \right|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt} \left(\frac{\partial I}{\partial \dot{x}} \right) v dt. \tag{13}$$

Since v vanishes at the endpoints this implies

$$0 = \int_{t_0}^{t_1} \left[\frac{\partial I}{\partial x} - \frac{d}{dt} \left(\frac{\partial I}{\partial \dot{x}} \right) \right] v dt. \tag{14}$$

If the above equality is to hold for every allowable v then it can be shown that

$$\frac{\partial I}{\partial x} - \frac{d}{dt} \left(\frac{\partial I}{\partial \dot{x}} \right) = 0 \text{ for every } t \in [x_0, x_1]. \tag{15}$$

The proof involves noting that if the above equation (15) were non-zero at some point, then by continuity it is non-zero on a neighbourhood of that point. Then an allowable function v is constructed that is non-zero on that neighbourhood. The result is called the *fundamental lemma of the calculus of variations.* The above ordinary second-order differential equation (15) is called *Euler's differential equation* or the *Euler condition* because it was discovered by Euler in 1744. Solutions to the Euler condition are called *extremals.* Note that they can be maxima or minima. Of course solutions to the Euler condition are maxima only if the above problem has an interior solution. I will not be concerned about corner solutions here.

[5] Recall that the formula for integration by parts is

$$\int_a^b F(x) g(x) dx = F(x) G(x)|_a^b - \int_a^b G(x) f(x) dx,$$

where $F'(x) = f(x)$ and $G'(x) = g(x)$.

2.2 Examples

Example 1.

$$\min_{x} \int_0^5 \left(2\dot{x}^2 + 3x\right) dt \text{ subject to } x(0) = 0,\ x(5) = 10. \tag{16}$$

The Euler equation is $\ddot{x} = 3/4$. Using the endpoints, $x = (t + 3t^2)/8$.

Example 2.

$$\max_{k} \int_0^T e^{-\delta t} \ln c(t)\, dt \text{ subject to } c(t) + \dot{k}(t) = rk(t) \text{ and } k(0) = k_0,\ k(T) = 0. \tag{17}$$

The Euler equations are the constraint and $\dot{c}(t)/c(t) = r - \delta$. Using the initial condition, these equations can be solved to find $c(t) = c_0 \exp[(r-\delta)t]$ and $k(t) = (k_0 - c_0/\delta)\exp(rt) + (c_0/\delta)\exp[(r-\delta)t]$, where c_0 is a constant. Using the terminal condition yields $c_0 = \delta k_0 / [1 - \exp(-\delta T)]$.

2.3 Second-order conditions

When solving static optimisation problems, it was necessary but not sufficient for an interior optimum that a candidate solution solved the first-order condition. The next step was to find the second-order conditions. Here, this is less convenient. However, if I is concave in its first two arguments, it is easy to show that this will suffice.

Suppose that $I(x(t), \dot{x}(t), t)$ is concave in its first two arguments and that $x^*(t)$ satisfies the Euler equation. Let $z(t)$ be defined as before. Then

$$\int_{t_0}^{t_1} I(z, \dot{z}, t)\, dt - \int_{t_0}^{t_1} I(x^*, \dot{x}^*, t)\, dt \leq \int_{t_0}^{t_1} \left[(z - x^*) \frac{\partial I(x^*, \dot{x}^*, t)}{\partial x} + (\dot{z} - \dot{x}^*) \frac{\partial I(x^*, \dot{x}^*, t)}{\partial \dot{x}}\right] dt \tag{18}$$

by concavity.[6] The right-hand side is equal to

$$\epsilon \int_{t_0}^{t_1} \left[v \frac{\partial I(x^*, \dot{x}^*, t)}{\partial x} + \dot{v} \frac{\partial I(x^*, \dot{x}^*, t)}{\partial \dot{x}}\right] = \epsilon \int_{t_0}^{t_1} \left[\frac{\partial I}{\partial x} - \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{x}}\right)\right] v dt = 0. \tag{19}$$

For many problems however this concavity requirement is too strong. By equation (12)

$$J''(0) = \int_{t_0}^{t_1} \left(\frac{\partial^2 I}{\partial x^2} v^2 + \frac{\partial^2 I}{\partial x \partial \dot{x}} v\dot{v} + \frac{\partial^2 I}{\partial \dot{x}^2} \dot{v}^2\right) dt. \tag{20}$$

[6] This property of concavity is discussed further in Lecture 3.

Integrating the middle term by parts and recalling that v vanishes at the end points yields

$$J''(0) = \int_{t_0}^{t_1} \left[\left(\frac{\partial^2 I}{\partial x^2} - \frac{d}{dt} \frac{\partial^2 I}{\partial x \partial \dot{x}} \right) v^2 + \frac{\partial^2 I}{\partial \dot{x}^2} \dot{v}^2 \right] dt. \tag{21}$$

It can be shown that a necessary condition for the right-hand side of equation (21) to be non-positive for all admissible v is that $\partial^2 I / \partial \dot{x}^2 \leq 0$. This is called the *Legendre condition.* The sign is reversed for a minimisation problem.

To summarise, the Euler equation and I concave in its first two arguments constitute necessary and sufficient conditions for an interior maximum. The Euler equation and the Legendre condition are necessary conditions but not sufficient conditions.

Example 3. $I = axt - b\dot{x}^2$. The Euler equation is $at + 2b\ddot{x} = 0$. The Legendre condition requires $b > 0$.

2.4 Extensions

The problems that economists consider usually have continuous solutions, but other sorts of problems have piecewise continuous solutions and so the theory of the calculus of variations was extended to deal with these problems. For other types problems $x(t_1)$ can be chosen by the planner or both t_1 and $x(t_1)$ can be chosen by the planner, perhaps subject to some constraints. It might also be the case that the value of the objective function depends on the terminal position as well as the path. The theory was also developed to deal with these cases. It can also be extended to problems of finding more than one path. The calculus of variations approach can handle certain types of constraints. The following constraints can all be handled with a Lagrangian approach:

Example 1. $\int_{t_0}^{t_1} f(x, \dot{x}, t)\, dt = c$.

Example 2. $F(x, \dot{x}, t) = 0$

Example 3. $F(x, \dot{x}, t) \leq 0$

3 Solving the Optimal Growth Model

The Euler equations for the optimal growth problem (8) are

$$\frac{\dot{c}}{c} = \frac{f'(k) - u - n - \delta}{\sigma(c)} \text{ where } \sigma(c) := -\frac{U''(c)\, c}{U'(c)} \tag{22}$$

$$\dot{k} = f(k) - (u + n) k - c. \tag{23}$$

3.1 A phase diagram approach

The properties of equilibrium can be analysed with a diagram in the (k, c) plane. The first step is to graph the set of points (the *isocline*) where $\dot{c} = 0$. This condition is satisfied if $f'(k) = u + n + \delta$. This is depicted in the Figure 4 below.

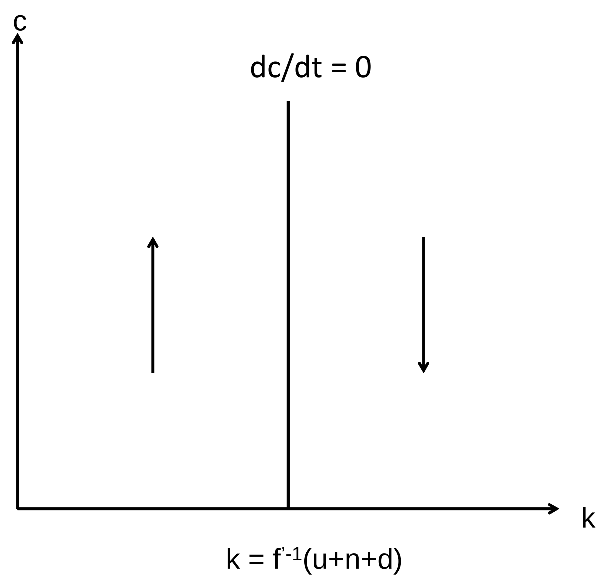

Figure 4. Isocline for c

If k is too large to satisfy the condition, then diminishing marginal productivity implies that $f'(k) < u + n + \delta$ and consumption is falling. This is indicated by the downward arrow to the right of the vertical line where $\dot{c} = 0$. Likewise, if k is too small to satisfy the condition then consumption is rising and this is shown by the upward arrow.

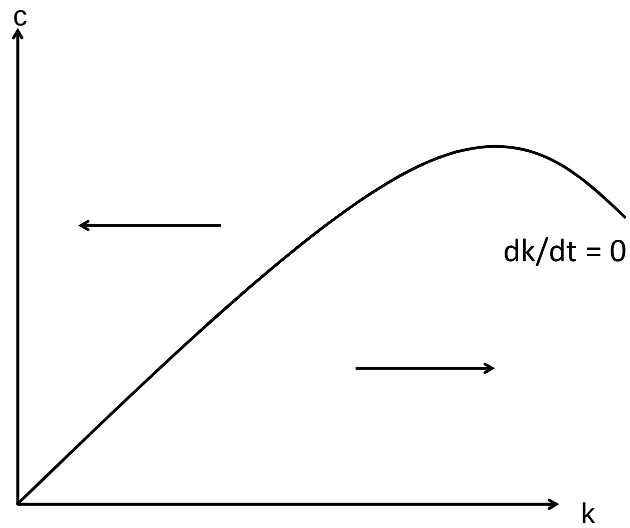

Figure 5. Isocline for k

The next step is to graph the set of points such that $\dot{k} = 0$. This occurs when $c = f(k) - (u + n)k$ and is depicted in the Figure 5 above. When $c > f(k) - (u + n)k$, then capital is falling. This is indicated by the rightward arrow above the $\dot{k} = 0$ curve. When $c < f(k) - (u + n)k$, then capital is rising. This is indicated by the rightward arrow above the $\dot{k} = 0$ curve.

Putting the two curves together in one plot yields Figure 6, below.

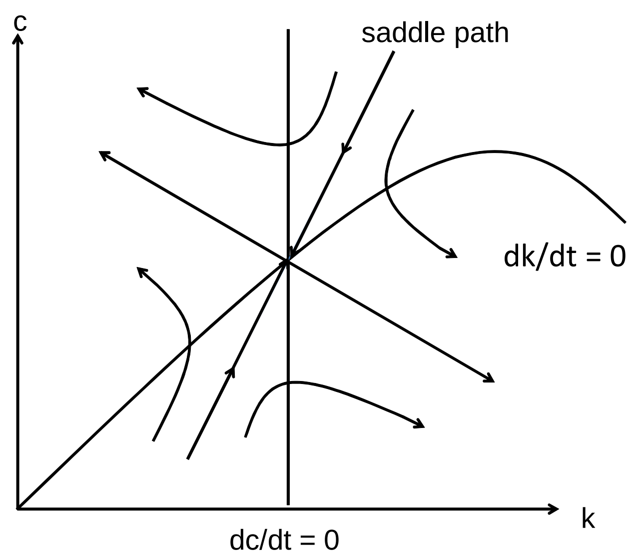

Figure 6. Phase Diagram

The two points intersect at the steady state (k_s, c_s) that satisfies

$$f'(k_s) = u + n + \delta \quad (24)$$

$$c_s = f(k_s) - (u+n)\,k_s. \quad (25)$$

Note that k_s is to the left of the capital stock that maximises consumption.

The initial capital stock is given but there is no terminal condition. This means that the planner can choose c_0. Suppose that $k_0 < k_s$. If c_0 is chosen to be in the northwest quadrant of the diagram in Figure 6 then consumption will continue to rise and capital will fall, becoming negative. This is not admissible. Suppose that $k_0 > k_s$. If c_0 is chosen to be in the southeast quadrant of the diagram then capital will continue to rise and consumption will fall, becoming negative. This is not admissible. Thus, if $k_0 < k_s$ then initial consumption must be in the southwest quadrant and if $k_0 > k_s$ consumption must be in the northeast quadrant. It turns out that in a neighbourhood around the steady state there is exactly one path that extends through the southwest and northeast quadrants and goes to the steady state. This path is knows as the *saddle path* or the *stable path*. Every other path that starts in the southwest or northeast quadrants goes into the northwest or southeast quadrants and is not admissible. This can be seen algebraically in the region around the steady state.

3.2 An analytical approach

Using equations (22) and (23), find a Taylor expansion around the steady state

$$\dot{c} = \frac{f''(k_s)}{\sigma(c_s)} c_s (k - k_s) \quad (26)$$

$$\dot{k} = \delta(k - k_s) - (c - c_s). \quad (27)$$

The general form of a solution to a pair of linear differential equations such as this is

$$c - c_s = a_0 \exp(\gamma_0 t) + a_1 \exp(\gamma_1 t) \quad (28)$$

$$k - k_s = b_0 \exp(\gamma_0 t) + b_1 \exp(\gamma_1 t). \quad (29)$$

Differentiationg equations (28) and (29) yields

$$\dot{c} = a_0\gamma_0 \exp(\gamma_0 t) + a_1\gamma_1 \exp(\gamma_1 t) \quad (30)$$

$$\dot{k} = b_0\gamma_0 \exp(\gamma_0 t) + b_1\gamma_1 \exp(\gamma_1 t). \quad (31)$$

Substituting equations (28) - (29) into equations (26) and (27) yields

$$a_0\gamma_0 \exp(\gamma_0 t) + a_1\gamma_1 \exp(\gamma_1 t) = \frac{f''(k_s)}{\sigma(c_s)} c_s [b_0 \exp(\gamma_0 t) + b_1 \exp(\gamma_1 t)] \quad (32)$$

$$b_0\gamma_0 \exp(\gamma_0 t) + b_1\gamma_1 \exp(\gamma_1 t) = \delta [b_0 \exp(\gamma_0 t) + b_1 \exp(\gamma_1 t)] - [a_0 \exp(\gamma_0 t) + a_1 \exp(\gamma_1 t)]. \quad (33)$$

Thus

$$a_0\gamma_0 = \frac{f''(k_s)}{\sigma(c_s)} c_s b_0 \text{ and } a_1\gamma_1 = \frac{f''(k_s)}{\sigma(c_s)} c_s b_1 \quad (34)$$

$$a_0 = \delta b_0 - b_0\gamma_0 \text{ and } a_1 = \delta b_1 - b_1\gamma_1. \quad (35)$$

Solving (34) and (35) for the γ_0 and γ_1 yields

$$\gamma_0 = \frac{1}{2}\left[\delta - \sqrt{\delta^2 - \frac{4c_s f''(k_s)}{\sigma(c_s)}}\right], \gamma_1 = \frac{1}{2}\left[\delta + \sqrt{\delta^2 - \frac{4c_s f''(k_s)}{\sigma(c_s)}}\right]. \quad (36)$$

By the equations in (36),

$$\gamma_0 < 0 < \gamma_1. \quad (37)$$

This implies that the steady state is (locally) a saddle point. That is, there is a unique path that approaches the steady state; all other paths diverge. To be on this unique saddle path, the coefficients on $\exp(\gamma_1 t)$, that is a_1 and b_1, need to be zero. So, using the initial condition, equations (28) and (29) become

$$c - c_s = a_0 \exp(\gamma_0 t) \quad (38)$$

$$k - k_s = b_0 \exp(\gamma_0 t). \quad (39)$$

Substituting in the initial condition and equation (35) yields

$$c - c_s = (\delta - \gamma_0)(k_0 - k_s) \exp(\gamma_0 t) \quad (40)$$

$$k - k_s = (k_0 - k_s) \exp(\gamma_0 t). \quad (41)$$

By equations (40) and (41) it can be seen that (in the area around the steady state) the capital-labour ratio is rising if and only if the initial capital-labour ratio is less than the steady-state capital-labour ratio. As $\delta > \gamma_0$ (by equation (36)), consumption is rising if and only if the initial capital-labour ration is less than the steady state capital labour ratio.

Equations (40) and (41) imply that

$$c = (\delta - \gamma_0)(k - k_s) + c_s. \tag{42}$$

This is the formula for the saddle path. As $\delta - \gamma_0 > 0$, the saddle path is upward sloping.

3.3 A pair of linear differential equations

Many economic models have equilibria characterised (at least locally) by a pair of first-order differential equations. These can be written as

$$\dot{x} = a_{11}x + a_{12}y + b_1 \tag{43}$$

$$\dot{y} = a_{21}x + a_{22}y + b_2. \tag{44}$$

Evaluating (43) and (44) at a steady state yields

$$0 = a_{11}x_s + a_{12}y_s + b_1 \tag{45}$$

$$0 = a_{21}x_s + a_{22}y_s + b_2. \tag{46}$$

Subtracting the left- and right-hand sides of equations (45) and (46) from the left- and right-hand sides of equations (43) and (44) and writing the result in matrix form yields

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x - x_s \\ y - y_s \end{bmatrix}, \text{ where } A := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{47}$$

Conjecture that the solution to equation (47) has the form $x - x_s = c_0 v_0 \exp(r_0 t) + c_1 v_1 \exp(r_1 t)$ and $y - y_s = c_0 \exp(r_0 t) + c_1 \exp(r_1 t)$.[7] Thus, by equation (47)

$$\begin{bmatrix} c_0 v_0 r_0 \exp(r_0 t) + c_1 v_1 r_1 \exp(r_1 t) \\ c_0 r_0 \exp(r_0 t) + c_1 r_1 \exp(r_1 t) \end{bmatrix} = A \begin{bmatrix} c_0 v_0 \exp(r_0 t) + c_1 v_1 \exp(r_1 t) \\ c_0 \exp(r_0 t) + c_1 \exp(r_1 t) \end{bmatrix}. \tag{48}$$

[7] This is equivalent to assuming that they have the form the form $x - x_s = c_0 \exp(r_0 t) + c_1 \exp(r_1 t)$ and $y - y_s = d_0 \exp(r_0 t) + d_1 \exp(r_1 t)$ but will turn out to be more convenient.

Since his has to be true no matter what t is this implies that,

$$r_i \begin{bmatrix} v_i \\ 1 \end{bmatrix} = A \begin{bmatrix} v_i \\ 1 \end{bmatrix}, \ i = 0, 1. \tag{49}$$

Thus, $r_i v_i = a_{11} v_i + a_{12}$ and $r_i = a_{21} v_i + a_{22}$, $i = 0, 1$. Thus, $-a_{12} = (a_{11} - r_i) v_i$ and $-(a_{22} - r_i) = a_{21} v_i$, $i = 0, 1$. Thus, $(a_{11} - r_i)(a_{22} - r_i) - a_{12} a_{21} = 0_i$, $i = 0, 1$. Thus,

$$\det \begin{bmatrix} a_{11} - r_i & a_{12} \\ a_{22} & a_{22} - r_i \end{bmatrix} = 0. \tag{50}$$

Thus, the roots r_1 and r_2 are the eigenvalues (or characteristic roots) of the matrix A in equation (47). If both of these roots are positive then x and y go to plus or minus infinity and the system is not stable; if both roots are negative then x and y go to their stationary their stationary points and the system is stable; if one root is positive and one root is negative the positive root eventually dominates and x and y go to plus or minus infinity and the system is not stable. The stationary point in this case is known as a saddle point.

By equation (49), the vector $\begin{bmatrix} v_i & 1 \end{bmatrix}^T$ satisfies

$$\begin{bmatrix} a_{11} - r_i & a_{12} \\ a_{22} & a_{22} - r_i \end{bmatrix} \begin{bmatrix} v_i \\ 1 \end{bmatrix} = 0, \ i = 0, 1. \tag{51}$$

Thus, $\begin{bmatrix} v_i & 1 \end{bmatrix}^T$ is the eigenvector or characteristic vector associated with root r_i, $i = 0, 1$. Note that only two of the four equations in (50) are independent and they determine v_i, $i = 0, 1$.

If both roots r_0 and r_1 are negative then the two initial conditions pin down the solution. Evaluating the solution at $t = 0$ we have $x_0 - x_s = c_0 v_0 + c_1 v_1$ and $y_0 - y_s = c_0 + c_1$. These tow equations can be solved to find c_0 and c_1. Suppose that one root is strictly positive and one root is strictly negative and let r_1 be the larger root. To find the saddle path set $c_2 = 0$ and use equation (51) to find v_1. If we are given the initial value for y we can evaluate the solution at $t = 0$ to find $y_0 - y_s = c_0$. Then the solution is $x - x_s = (y_0 - y_s) v_0 \exp(r_0 t)$ and $y - y_s = (y_0 - y_s) \exp(r_0 t)$. By these two equations, $y - y_s = v_0 (x - x_s)$. This equation gives y as a function of x on the saddle path.

3.4 Comparative Dynamics

It is typical in economic models to perform the following experiment. Suppose that the economy is at a steady state and then there is an unexpected change in one of the exogenous variables. This change must be viewed as small if one is looking at an approximation around a steady state. Then one asks, what does the new steady state look like and what does the path to it look like? An underlying assumption in performing this experiment is that the economic agents in the model have perfect foresight except at the one point where they are surprised by the change in the exogenous variable.

This section considers a surprise increase in δ. By equations (26) and (27)

$$\frac{\partial k_s}{\partial \delta} = \frac{1}{f''(k_s)} < 0, \ \frac{\partial c_s}{\partial \delta} = \frac{\delta}{f''(k_s)} < 0. \tag{52}$$

Both the steady state capital-labour ratio and consumption fall.

Graphically, the $\dot{k} = 0$ isocline (found from equation (23)) is unaffected by δ but the $\dot{c} = 0$ isocline (found from equation (22)) shifts leftward. The result can be seen in Figure 7 below

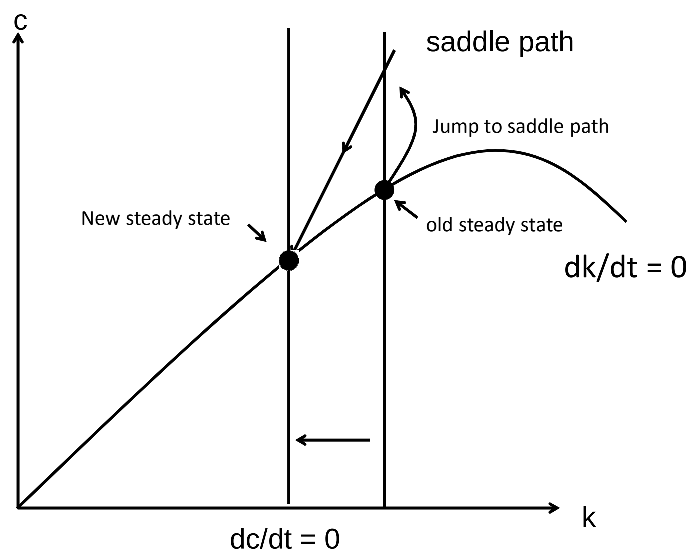

Figure 7. Change in δ

The new steady state is to the southwest of the old one. From the old one, consumption jumps up to put the economy on the stable path. Then, consumption and the capital-labour ratio decline over time. This makes sense, with a higher discount rate, current consumption is valued more so it jumps up.

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