# Lecture 2: The Representative Agent Model

This lecture considers two simple representative agent models with money in the utility function: the first is a small open economy version and the second is a closed economy version. An optimal control approach is used to solve these models. Before setting up the models I will review how this works.

## 1 Optimal control theory

The last lecture considered a classical calculus of variations approach to optimisation. The modern theory of optimal control, developed by L. S. Pontryagin and his students in the late 1950s, allows for inequality constraints and is based on a different way of formulating problems that is often more convenient.

#### 1.0.1 The optimal control problem

Optimal control was developed to control optimally over time some system such as a rocket. It is assumed that the way that the system changes over time can be described by the change over time in a finite number of variables called *state variables*. It is also supposed that there exists a finite set of variables called *control variables* such that if the paths of the control variables are specified this determines the path of the state variables. In particular, if the state variables are denoted by $x$ and the control variables are denoted by $y$ then the behaviour of the system is described by a set of first-order differential equations $\dot{x} = f(x, y, t)$. The control variables are not unrestricted but must remain in some admissible set.

To keep the notation down, it is supposed that there is one state variable and one control variable. Consider the simple control problem

$$\begin{aligned} \max J &= \int_{t_0}^{t_1} I(x, y, t)\, dt \text{ subject to} \\ \dot{x} &= f(x, y, t) \\ & t_0,\ t_1,\ x(t_0) \text{ given.} \end{aligned} \tag{1}$$

The functions $f$ and $I$ are assumed to be continuously differentiable in their three arguments. The control variable is required to be piecewise continuous.

Note that the calculus of variations problem

$$\max J = \int_{t_0}^{t_1} I(x, \dot{x}, t)\, dt \text{ where } t_0,\ t_1,\ x(t_0) \text{ are given} \tag{2}$$

from the last lecture can be transformed into a control problem by letting $y = \dot{x}$. Then

$$\max J = \int_{t_0}^{t_1} I(x, y, t)\, dt \text{ subject to } \dot{x} = y \text{ where } t_0,\ t_1,\ x(t_0) \text{ are given.} \tag{3}$$

### 1.0.2 Necessary conditions

The outcome of the research in optimal control are theorems that state the necessary conditions, called *canonical equations*, for optimality. This section considers the sufficient conditions for the simple control problem (1). No attempt is made to reproduce a proof that the conditions are sufficient. Instead, a heuristic way to find them is presented.

In the calculus of variations approach an optimal path $x^*$ was considered and comparison paths $z$ were defined. It was shown that the objective function was optimised at $z = x^*$. The obvious problem with doing the same thing here is that if one begins with an optimal path $y^*$ for the control and then defines comparison paths for the control denoted by $z$, it is unclear what the resulting paths for $x$ look like when $y = z$. However, by using a multiplier approach one can use a similar procedure even though the functional form of the comparison $x$ paths is unknown.

Define a Lagrangian using a new variable $\lambda(t)$ called a *costate variable* that is analogous to the Lagrange multiplier in a static optimisation problem:

$$J = \int_{t_0}^{t_1} \{I(x, y, t) + \lambda [f(x, y, t) - \dot{x}]\}\, dt. \tag{4}$$

Integrating by parts, equation (4) can be rewritten as

$$J = \int_{t_0}^{t_1} [I(x, y, t) + \lambda f(x, y, t)]\, dt - \int_{t_0}^{t_1} \lambda \dot{x} dt.$$

$$= \int_{t_0}^{t_1} [I(x, y, t) + \lambda f(x, y, t)]\, dt - \lambda x|_{t_0}^{t_1} + \int_{t_0}^{t_1} \dot{\lambda} x dt.$$

$$= \int_{t_0}^{t_1} \left[I(x, y, t) + \lambda f(x, y, t) + \dot{\lambda} x\right] dt - \lambda(t_1)\, x(t_1) + \lambda(t_0)\, x(t_0)\,. \tag{5}$$

The expression

$$H(x, y, \lambda, t) := I(x, y, t) + \lambda f\ (x, y, t) \tag{6}$$

is known as the *Hamiltonian function.* Rewriting equation (5) using the definition in (6) yields

$$J = \int_{t_0}^{t_1} \left[H(x, y, \lambda, t) + \dot{\lambda} x\right] dt - \lambda(t_1)\, x(t_1) + \lambda(t_0)\, x(t_0)\,. \tag{7}$$

In the last lecture the Euler equation was found by considering a comparison function $x^* + \epsilon v$, where $v$ was a fixed function and $\epsilon$ was a parameter. Here a comparison function of the control $y^* + \epsilon v$ is considered. Let $z(t, \epsilon)$ be the value of the state variable when the control variable is $y^* + \epsilon v$. We have $x^*(t) = z(t, 0)$ and $z(t_0, \epsilon) = x_0$.

Evaluating the right-hand side of equation (7) at the comparison function gives $J$ as a function of $\epsilon$:

$$J(\epsilon) = \int_{t_0}^{t_1} \left[ H(z(t, \epsilon), y^* + \epsilon v, \lambda, t) + \dot{\lambda} z(t, \epsilon) \right] dt - \lambda(t_1) z(t_1, \epsilon) + \lambda(t_0) x_0. \tag{8}$$

Supposing that $z$ is differentiable, equation (8) implies

$$J'(0) = \int_{t_0}^{t_1} \left\{ H_y(x^*, y^*, \lambda, t) v + \left[ H_x(x^*, y^*, \lambda, t) + \dot{\lambda} \right] z_\epsilon(t, 0) \right\} dt - \lambda(t_1) z_\epsilon(t_1, 0). \tag{9}$$

We don't know what $z_\epsilon(t, 0)$ is equal to, but suppose that $\dot{\lambda} = -H_x(x^*, y^*, \lambda, t)$ and $\lambda(t_1) = 0$. Then $J'(0) = 0$ as long as

$$\int_{t_0}^{t_1} H_y(x^*, y^*, \lambda, t) v dt = 0. \tag{10}$$

Let $v = H_y$ and then by (10) this requires that $H_y(x^*, y^*, \lambda, t) = 0$.

To summarise the results of this informal approach, if $(x^*, y^*)$ maximises $J$ then

$$H_y = 0 \tag{11}$$

$$-H_x = \dot{\lambda} \tag{12}$$

$$H_\lambda = \dot{x} \tag{13}$$

and $\lambda(t_1) = 0$. At each point in time the control variable $y$ must maximise $H$: this is the *maximum principle*. For this to be true it is necessary that $H_{yy} < 0$.

In principle, one can solve equations (11) - (13) by solving equation (11) to find $y = y(x, \lambda, t)$ and substituting this into equations (12) and (13). One then has two first-order differential equations in two unknowns: the state variable and the costate variable.[1] If these equations are autonomous (that is, if they do not depend directly on time), then they can be analysed with a phase diagram approach. In economics it is typical that the steady state is a saddle point. The initial value of the state variable is typically given and there is typically a requirement – either as part of the optimality conditions or a result of the preferences of the person setting forth the problem – that the system remains on the saddle path. While the state variable

[1] Equivalently one can usually rewrite (11) - (13) to find two first-order conditions in the state variable and the control variable.

cannot jump, the costate variable can and it does so to be on the saddle path. Thus, the outcome is uniquely determined.

For most simple economic problems it is a matter of taste whether one uses the calculations of variations approach or the optimal control approach as the outcomes are equivalent. However, the solution to the calculus of variations problem is a second-order differential equation, which is usually converted to two first-order differential equations. The solution to the optimal control problem is naturally two first-order differential equations.

# 2 Two Representative Agent Models

In this section the maximum principle is used to solve two representative agent models.

## 2.1 Small open economy representative agent model with money in the utility function

A small open economy is inhabited by a representative household and the government. There is a single good and the home country takes the foreign currency price of the good and the foreign interest rate as given. The inhabitants have perfect foresight.

### 2.1.1 Two parity conditions

**purchasing power parity** It is assumed that *purchasing power parity* holds. Purchasing power parity is a generalisation of the *law of one price.* The law of one price is a rule of thumb that says that it costs the same amount to buy the single home good whether you buy it with home money or convert the home money into foreign money and buy the good with the foreign money. That is, if a cashmere sweater sells for £200 in Harrods and the pound is trading at two dollars per pound then the same sweater should sell for $400 at Saks Fifth Avenue. This law is an imperfect rule of thumb, but it is a convenient modeling assumption. With one good – as is the case here – the law of one price and purchasing power parity amount to the same thing and $P = eP^*$, where $P$ is the home currency price of the single good, $P^*$ is the foreign currency price of the good and $e$ is the exchange rate expressed as the home currency price of the foreign currency. When there are multiple goods – as is the case in the real world – purchasing power parity is the extension of the law of one price to price levels. It can be expressed as $P = eP^*$, where the prices are consumer price indices. This is called absolute purchasing power parity.

Recall – this will be useful for this lecture – that if you have an expression such as $z_t = x_t y_t^{\alpha}/u_t$, then it is often convenient to differentiate it as $\dot{z}_t/z_t = \dot{x}_t/x_t + \alpha\dot{y}_t/y_t - \dot{u}_t/u_t$. Applying this to $P = eP^*$ yields

$\dot{P}/P = \dot{e}/e + \dot{P}^*/P^*$. The terms $\dot{P}/P$ and $\dot{P}^*/P^*$ are home and foreign inflation, respectively, and it is typical to denote them by $\pi$ and $\pi^*$, respectively. Thus,

$$\pi = \pi^* + d, \tag{14}$$

where $d = \dot{e}/e$ is the rate of depreciation of the home currency. The expression in equation (14) is called (relative) purchasing power parity.[2] It is assumed here that the exogenous variable $\pi^*$ is a constant.

**uncovered interest parity** Capital is perfectly mobile and there is no uncertainty. Thus, with perfect foresight the instantaneous return on home-currency denominated bonds should equal the instantaneous return on foreign-currency denominated bonds. This parity condition is called *uncovered interest parity* and it requires that

$$i = i^* + d, \tag{15}$$

where $i$ and $i^*$ are the home- and foreign-currency nominal interest rates, respectively. As the home country is small it takes $i^*$as given and it is assumed here that this exogenous variable is a constant.

#### 2.1.2 The household's optimisation problem

Households have preferences defined over their consumption paths and the paths of their home real money balances. The household's objective function is

$$\int_0^\infty \exp\left(-\delta t\right)\left[u\left(c\right) + v\left(m\right)\right] dt, \;\; \delta > 0, \tag{16}$$

where $c$ is consumption and $m$ is home real balances. It is assumed that $u$ and $v$ have the usual nice properties.

It is assumed that the household is endowed with a constant flow $y$ of the good and pays a lump-sum tax $\tau$. In addition to saving home money, denoted by $M$, the household saves home- and foreign-currency bonds. By uncovered interest parity (15), the household is indifferent over different compositions of its bond portfolio. Let $A$ be the total value of the households assets, including bonds and money, in terms of the home currency. The household's budget constraint is then

$$c + \frac{\dot{A}}{P} = y - \tau + \frac{i\left(A - M\right)}{P}, \tag{17}$$

where $M_0$ and $A_0$ are given.

[2] Equation (14) is equivalent to $P = \alpha e P^*$, where $\alpha$ is any strictly positive constant.

We have $a = A/P$; hence $\dot{a}/a = \dot{A}/A - \dot{P}/P$. This implies $\dot{A}/P = \dot{a} + \pi a$. Rewriting the household budget constraint (17) yields

$$\dot{a} = y - \tau + (i - \pi) a - im - c. \tag{18}$$

Substituting the purchasing power parity (14) and uncovered interest parity (15) conditions into equation (18) yields

$$\dot{a} = y - \tau + (i^* - \pi^*) a - (i^* + d) m - c. \tag{19}$$

To solve the problem, set up the Hamiltonian

$$H = \exp(-\delta t) [u(c) + v(m)] + \lambda [y - \tau + (i^* - \pi^*) a - (i^* + d) m - c]. \tag{20}$$

The control variables are $c$ and $m$; $a$ is the state variable and $\lambda$ is the costate variable.

An interior maximum has

$$H_c = 0,\ H_m = 0,\ -H_a = \dot{\lambda},\ H_\lambda = \dot{a}. \tag{21}$$

Differentiating (20) yields

$$\exp(-\delta t) u'(c) = \lambda \tag{22}$$

$$\exp(-\delta t) v'(m) = \lambda (i^* + d) \tag{23}$$

$$-\lambda (i^* - \pi^*) = \dot{\lambda} \tag{24}$$

$$y - \tau + (i^* - \pi^*) a - (i^* + d) m - c = \dot{a}. \tag{25}$$

Let $\mu := \exp(\delta t) \lambda$ and thus $\dot{\mu} - \delta\mu = \exp(\delta t) \dot{\lambda}$. Rewriting (22) - (24) yields

$$u'(c) = \mu \tag{26}$$

$$v'(m) = \mu (i^* + d) \tag{27}$$

$$\mu (\delta - i^* + \pi^*) = \dot{\mu}. \tag{28}$$

In addition to the Euler equations (25) - (28), a terminal condition that rules out Ponzi games where the household borrows ever larger amounts is imposed. There is also a transversality condition that is part of the optimal conditions and says that the present discounted value of the household's savings should become non-positive as time goes to infinity. We will discuss these conditions at length next time but for now they are written in their usual form as

$$\lim_{t\to\infty} \mu m \exp\left[-\left(i^* - \pi^*\right) t\right] = \lim_{t\to\infty} \mu a \exp\left[-\left(i^* - \pi^*\right) t\right] = 0. \tag{29}$$

#### 2.1.3 The government's budget constraint

The government's instantaneous purchases, denoted by $g$, are assumed to be constant. To finance these purchases, as well as the interest on past borrowing, the government prints money, issues debt and imposes (constant) taxes. Its budget constraint satisfies

$$\frac{\dot{M}}{P} + \frac{\dot{B}}{P} + \tau = g + \frac{iB}{P}, \tag{30}$$

where $B$ is the nominal value in terms of home currency of the debt that it issues. It takes $M_0$ and $B_0$ as given.

As with the household's budget constraint, the governments' budget constraint (30) can be rewritten using purchasing power parity (14) and uncovered interest parity (15) as

$$\dot{m} + \dot{b} = g - \tau + \left(i^* - \pi^*\right) b - \left(\pi^* + d\right) m. \tag{31}$$

It is assumed that the government chooses a constant money growth rate $\dot{M}/M = z$. Using the purchasing power parity condition (14) yields $\dot{m}/m = z - \pi = z - \pi^* - d$. Thus

$$\dot{m} = \left(z - \pi^* - d\right) m. \tag{32}$$

The government must also satisfy a terminal condition:

$$\lim_{t\to\infty} \mu b \exp\left[-\left(i^* - \pi^*\right) t\right] = 0. \tag{33}$$

#### 2.1.4 Equilibrium

The dynamic efficiency condition (28) can be rewritten as :

$$\delta - \frac{\dot{\mu}}{\mu} = i^* - \pi^*. \tag{34}$$

This equation says that in equilibrium the rate of return on savings (the right-hand side) must equal the rate of return on consumption (the left-hand side). As can be seen from equation (34) the assumption that the economy is small and open and takes $i^* - \pi^*$ as a given constant implies that marginal utility, given by equation (26), goes to plus or minus infinity unless one makes the unappealing assumption that $\delta = i^* - \pi^*$and, thus, $\mu$ is a constant. We proceed to do this.

By equation (26), $c$ is a constant and equal to $u'^{-1}(\mu)$ and by equation (27), $d = v'(m)/\mu - i^*$. Thus by equation (32)

$$\dot{m} = \left[z - \pi^* - \frac{v'(m)}{\mu}\right] m. \tag{35}$$

It is easy to see that the differential equation (35) is not stable. If "bubbles" where the price level goes to infinity and real balances go to zero are ruled out, then it must be that $d = z + \delta$. Thus, by equations (26), (27) and (32)

$$\begin{aligned} c &= u'^{-1}(\mu) && (36)\\ m &= v'^{-1}(\mu(i^* + z + \delta)) && (37)\\ d &= z - \pi^* && (38) \end{aligned}$$

Using the result that consumption and real balances are constant and substituting equation (38) into the budget constraints (25) and (31) yields

$$\begin{aligned} \dot{a} - \delta a &= \tilde{Y} := \frac{y - \tau - (\delta + z) m - c}{\delta} && (39)\\ \dot{b} - \delta b &= \tilde{G} := \frac{g - \tau - zm}{\delta}. && (40) \end{aligned}$$

Integrating (39) and (40) implies

$$\begin{aligned} a &= -\tilde{Y} + \exp(\delta t)\left(\tilde{Y} + a_0\right) && (41)\\ b &= -\tilde{G} + \exp(\delta t)\left(\tilde{G} + b_0\right). && (42) \end{aligned}$$

Using equations (39) and (40), the transversality conditions (29) and (33) require that

$$\begin{aligned} a_0 &= -\tilde{Y} && (43)\\ b_0 &= -\tilde{G}. && (44) \end{aligned}$$

Substituting (43) and (44) into (41) and (42) yields

$$a = -\tilde{Y},\, b = -\tilde{G}. \tag{45}$$

By equation (44), $p_0 = -B_0/\tilde{G}$. This implies that $m = -M_0\tilde{G}/B_0$.This model has no dynamics. If there is an unforeseen disturbance, the path back to equilibrium is accomplished by a one-time jump in the price level.

## 2.2 Sidrauski's Model: Money in the Utility Function in a Closed Economy

Sidrauski's model is a closed economy where the household holds money and can accumulate capital.

### 2.2.1 Households

The household has preferences defined over paths of consumption and real balances given by

$$\max \int_0^\infty e^{-\delta t} \left[ u(c) + v(m) \right] dt, \;\; \delta > 0. \tag{46}$$

Its budget constraint is

$$c + \frac{\dot{M}}{P} + \dot{k} = f\ (k) + T. \tag{47}$$

Letting $a := m + k$, equation (47) can be rewritten as

$$\dot{a} = f\ (a - m) + T - \pi m - c. \tag{48}$$

Using equations (46) and (48) the Hamiltonian is

$$H = e^{-\delta t} \left[ u(c) + v(m) \right] + \lambda \left[ f\ (a - m) + T - \pi m - c \right]. \tag{49}$$

The Euler equations are the constraint (48) and

$$\begin{aligned} e^{-\delta t} u'(c) - \lambda &= 0 \\ e^{-\delta t} v'(m) - \lambda \left[ f'(a - m) + \pi \right] &= 0 \\ \lambda f'(a - m) + \dot{\lambda} &= 0 \end{aligned}$$

Letting $\mu e^{-\delta t} = \lambda$ these equations can be rewritten as

$$u'(c) = \mu \tag{50}$$

$$v'(m) = \mu \left[ f'(k) + \pi \right] \tag{51}$$

$$\dot{\mu} = \mu \left[ \delta - f'(k) \right]. \tag{52}$$

### 2.2.2 The government

The government's budget constraint is

$$\dot{m} = T - \pi m. \tag{53}$$

By equations (45) and (53) the economy-wide resource constraint is

$$\dot{k} = f\ (k) - c. \tag{54}$$

### 2.2.3 Equilibrium

By equation (50)

$$c = c\,(\mu) \text{ where } c'\,(\mu) = -\frac{c}{\sigma\mu},\ \sigma := -cu''/u'. \tag{55}$$

Substituting equation (55) into equation (54) yields

$$\dot{k} = f\ (k) - c\,(\mu)\,. \tag{56}$$

Linearising equations (52) and (56) around the steady state yields

$$\dot{\mu} = -\mu f\,''\,(k - k_s)\,,\ \dot{k} = \frac{c}{\sigma\mu}\,(\mu - \mu_s) + f\,'\,(k - k_s) \tag{57}$$

Writing equations (57) in matrix form yields

$$\begin{bmatrix} \dot{\mu} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} 0 & -\mu f\,'' \\ \frac{c}{\sigma\mu} & f\,' \end{bmatrix} \begin{bmatrix} \mu - \mu_s \\ k - k_s \end{bmatrix}. \tag{58}$$

The eigenvalues of the matrix in equation (58) are the real values

$$r_1 = \frac{f\,' - \sqrt{(f\,')^2 - 4\frac{f\,''c}{\sigma}}}{2} < 0 < \frac{f\,' + \sqrt{(f\,')^2 - 4\frac{f\,''c}{\sigma}}}{2} = r_2. \tag{59}$$

Thus, the steady state is a saddle point. The eigen vector associated with the smaller root is

$$\begin{bmatrix} -\frac{\mu f\,''}{r_1} \\ 1 \end{bmatrix}. \tag{60}$$

Thus, using the initial condition $k_0$,

$$\mu - \mu_s = -\frac{\mu f''(k - k_0)\, e^{-rt}}{r_1},\ k - k_s = (k - k_0)\, e^{-rt}. \tag{61}$$

Using equations (61), the equation for the saddle path is

$$\mu - \mu_s = \frac{-\mu f''(k - k_s)}{r_1}. \tag{62}$$

Note that an interesting thing about this version of Sidrauski's model is that money is *superneutral.* That is, the real variables in the model do not depend on the rate of monetary expansion. This would not be true if the utility function were not separable in $c$ and $m$.

Suppose that the rate of monetary expansion is $\dot{M}/M = z$. Then $\dot{m}/m = z - \pi$ and equation (51) implies

$$v'(m) = \mu \left[ f'(k) + z - \frac{\dot{m}}{m} \right]. \tag{63}$$

Rewriting equation (63) yields

$$\dot{m} = \left[ f'(k) + z - \frac{v'(m)}{\mu} \right] m. \tag{64}$$

This is differential equation is unstable. If bubbles are ruled out then

$$[f'(k) + z]\, \mu = v'(m). \tag{65}$$

## 2.3 References

Hadley, G. and M. C. Kemp, *Variational Methods in Economics*, Amsterdam, North-Holland, 1971, ch. 4.

Intriligator, Michael D., *Mathematical Optimization and Economic Theory*, Englewood Cliffs, NJ, Prentice-Hall, 1971, ch. 14.

Kamien, Morton I. and Nancy L. Schwartz, *Dynamic Optimisation: The Calculus of Variations and Optimal Control in Economics and Management*, Amsterdam, North-Holland, 1981, pp. 111 - 121.

Turnovsky, Stephen J., International Macroeconomic Dynamics, London, MIT Press, 1997, section 2.1 [Open-Economy representative agent model with money in the utility function.

Turnovsky, Stephen J., *Methods of Macroeconomic Dynamics*, London, MIT Press, 2nd edition, 2000, sections 8.1-2. [Representative agent model in a closed economy with production.]