MSc Financial Economics: International Finance

Bubbles in the Foreign Exchange Market

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Contents

1. Introduction

During the Bretton Woods era between the end of World War II and August 1971, the world was on a fixed exchange rate system and there was little international capital mobility. The primary determinant of the balance of payments was the trade balance. The exchange rate was viewed as the relative price of home and foreign goods. The equilibrium value of the exchange rate was one that led to trade balance.

In as world where capital is mobile across industrialised countries, this view is not appropriate and the exchange rate is better viewed as the relative price of different currencies or the relative price of financial assets denominated in different currencies. The model in this lecture is an example of the former approach. It is particularly useful for looking at how anticipated changes in the current and future values of the money supply affect the current exchange rate.

The model is of a small open economy with a single good. The home currency is held only by domestic residents and home residents do not hold foreign currency. Both residents of the home country and residents of the rest of the world hold home-currency and foreign-currency bonds. Output is assumed to be exogenous. Time occurs in discrete intervals.

2. The Money Market

Equilibrium in the money market is given by

$$M_t/P_t = YI_t^{-\alpha}, \alpha > 0 \tag{1}$$

where M_t is the money supply, P_t is the home-currency price of the good, Y is (constant) output or income and I_t is one plus the home-currency interest rate.

The above equation is similar to the LM curve of undergraduate macroeconomics courses. The right-hand side is the demand for real balances, which is assumed to be increasing in income and decreasing in the nominal interest rate. The parameter α is referred to as the semi-elasticity of money demand with respect to the interest rate and it measures how sensitive money demand is to the interest rate. The higher is α, the greater the sensitivity; as α goes to zero, money demand becomes insenstive to the interest rate. The particular functional form is chosen because it is especially tractable.

Take the logarithm of both sides of equation (1) and let small letters denote the logarithm of capital letters so that, for example, $x = \ln X$. We have

$$m_t - p_t = y - \alpha i_t. \tag{2}$$

2.1. Purchasing power parity. Assume that *purchasing power parity* holds. Let E_t be the exchange rate, or the price of the foreign currency in terms of the home currency and let P_t^* be the foreign currency price of the single good. The assumption of purchasing power parity here means that it costs the same amount to buy the good whether one buys it directly, paying P_t units of home currency, or whether one buys P_t^* units of the foreign currency with $E_t P_t^*$ units of the home currency and buys the good with the foreign currency. Thus,

$$P_t = E_t P_t^*. \tag{3}$$

As the country is small, we takes the foreign currency price of the good as given or exogenous. To save on notation we will assume that it is constant. We can then pick the units in which the foreign good is measured so that the foreign currency price equals one. Then, taking logarithms of both sides of the above equation we have

$$p_t = e_t. \tag{4}$$

2.2. Uncovered interest parity. The fundamental assumption of the monetary approach is that interest-bearing financial assets denominated in different currencies are *perfect substitutes.* This means that investors care only about the expected return on assets and not about how risky they are. Let I_t^* be one plus the interest rate on foreign-currency denominated assets. The small-country assumption means that this variable is also exogenous. We will further assume that it is a constant.

An investor can take one unit of home currency and invest it in home-currency bonds. At the end of the period he will have I_t units of home currency. Or, he can convert one unit of home currency into $1/E_t$ units of foreign currency and then invest the foreign currency in foreign-currency bonds. At the end of the period he will have I^*/E_t units of foreign currency, which he can then trade back for $E_{t+1}I^*/E_t$ units of home currency. Thus, for investors to be willing to hold both home-currency bonds and foreign-currency bonds, on average I_t must be equal to $E_{t+1}I^*/E_t$. This condition is called *uncovered interest parity.*

2.3. Perfect foresight. We assume that there is no uncertainty in the model. Market participants know the entire history and the entire future path of the money supply. Thus, they can solve for the path of the exchange rate. Hence, we assume that market participants can correctly predict the future exchange rate. This assumption is called *perfect foresight.* In this case uncovered interest parity implies that

$$I_t = E_{t+1}I^*/E_t. \tag{5}$$

Taking logarithims of both sides yields

$$i_t = i^* + e_{t+1} - e_t. \tag{6}$$

Substituting equations (4) and (6) into equation (2) yields

$$m_t - e_t = y^* - \alpha(e_{t+1} - e_t). \tag{7}$$

This summarises the model. It is a first-order linear difference equation with a constant coefficient.

We could rewrite this as

$$e_{t+1} = a + cm_t + be_t, \tag{8}$$

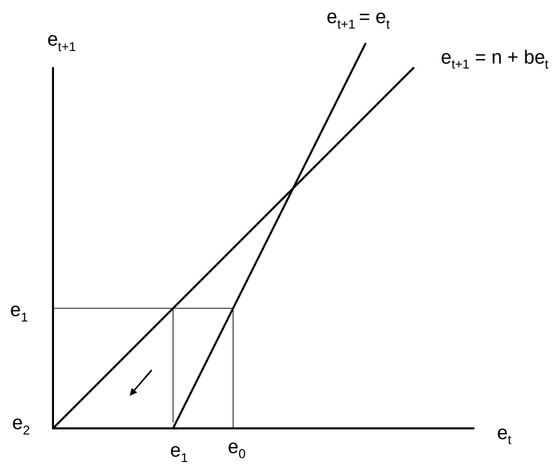
where the values of a, b and c can be found from equation (7).

2.4. The equilibrium is not stable. Typically in mathematics books, equations such as equation (8) would be solved *backwards* to find e_t. This would work as follows. By equation (8), the time-$(t+1)$ exchange rate depends on the time-t money supply and the time-t exchange rate. Also by equation (8), the time-t exchange rate depends on the time-$t-1$ money supply and on the time-$(t-1)$ exchange rate. Hence, the time-$t+1$ exchange rate can be expressed as a function of the time-t money supply, the time-$(t-1)$ money supply and the time-$(t-1)$ exchange rate. The time-$(t-1)$ exchange rate depends on the time-$(t-2)$ money supply and the time-$(t-2)$ exchange rate. Thus, the time-$t+1$ exchange rate can be expressed as a function of the time-t, the time-$(t-1)$ and the time-$(t-2)$ money supplies and the time-$(t-2)$ exchange rate. Continuing in this fashion, the time-$t+1$ exchange rate depends on the money supply for periods zero through t and on e_0.

The equation is the discrete-time analogue of a differential equation. Recall that there are a continuum of a solutions to a first-order differential equation and the way we typically pick out one of them is to say that we know the value of the solution as some point. Solving the above equation backwards and using some historical value to pin down the solution is the analogue to this.

Consider a simple example of equation (8) where m_t is constant and $n \equiv a + cm$. Then we can graph the equation. It will turn out that everything interesting depends on the sign and magnitude of the variable b.

The figure below depicts the case of $1 < b$.



I have graphed the time-t value of the exchange rate on the horizontal axis and the time-$(t+1)$ value on the vertical axis. The two upward sloping lines are $e_{t+1} = n + be_t$ and $e_{t+1} = e_t$. The assumption that $b > 1$ ensures that the slope of the former exceeds the slope of the latter. The point where the two curves intersect is the stationary point. There is equilibrium and the exchange rate is constant. Thus, if the economy is ever at this point, the exchange rate does not change.

Imagine we started off at some initial value e_0. Then we can find e_1 on the vertical axis by using the curve $e_1 = n + be_0$.Once we have e_1 on the vertical axis we can find e_1 on the horizontal axis by using the curve $e_1 = e_0$. Repeating the process, it is clear from the picture that the exchange rate goes off monotonically to minus infinity.[1] If we had chosen an e_0 greater than the stationary point, then we would have found that the exchange rate goes off to plus infinity. Only when we start at the intersection of the two curves does this not happen.[2]

Looking at equation (7), we see that n in equation (8) must be $(1+\alpha)/\alpha > 1$. Our model of the exchange rate is not stable!

The above result is typical in models of financial assets; it turns out to be sensible. Consider equation (7) again. The current exchange rate depends on the current state of the economy, summarised by the money supply, and the expected depreciation of the exchange rate. People have perfect foresight; hence, the

[1] The solution is monotonic if it is always increasing or always decreasing.
[2] Other cases are $0 < b < 1$, which is monotonic and stable (the exchange rate goes to the stationary point), $-1 < b < 0$, which is stable and has oscillations and $b < -1$ which has explosive oscillations. Try drawing the pictures to see for yourself.

expected future exchange rate is the actual future exchange rate. Thus, the current exchange rate depends on the current state of the economy and next period's exchange rate. Similarly, next period's exchange rate depends on next-period's state of the economy and the two-period-ahead exchange rate. Thus, the current exchange rate depends on the current state of the economy, next-period's state of the economy and the two-period-ahead exchange rate. Continuuing with this logic, the current exchange rate depends on the current and all future states of the economy and the exchange rate infinitely far out into the future.

The graphical solution used a different approach. With our graph, we tried to find the current exchange rate, given the past exchange rate and the current state of the economy. This was how people in the sciences would typically proceed and how mathematics books tend to view the problem. If we think about the intuition, there is something odd about this. First, as the above intuition suggests, it does not seem sensible to view the current exchange rate as depending on the past. Second, in the graphical example, the money supply is constant. So, why should the exchange rate vary over time? Does it not seem more reasonable that it would be constant as well? What if instead of trying to solve the equation backwards – as we did with the graphical approach – we solve it forward, as the thought experiment in the previous paragraph suggests.

In the graphical solution, we saw that the path of the exchange rate depended on the initial value of the exchange rate. Relating this to the math, first-order difference equations have an infinite number of solutions, just as first-order differential equations do. When we solve a difference or differential equation backwards we pick out a particular solution by saying that we know its value at some point. Typically, we would say we know what the value is at zero. This is called a *boundary condition*. When we solve the model forward we will still have an infinite number of solutions. But, now we cannot pick out a path by saying we know the initial value. From the thought experiment, the current exchange rate is going to depend on the current and all future values of the money supply and on the exchange rate infinitely far out into the future. Thus, our boundary condition is going to be a condition on this infinitely far in the future exchange rate. We will assume that it is unreasonable for the exchange rate to go to plus or minus infinity when it is not warranted by the fundamentals. It will turn on that there is only one path that satisfies this. In the above picture, this amounts to saying that the only reasonable equilibrium when the money supply is constant is the steady-state equilibrium where the exchange rate is constant as well. In the next section, I show how to solve the first-order linear difference equation forward and how to pick out the solution we want.

3. Solving for the Path of the Exchange Rate

In this section I consider two methods for solving for the path of the exchange rate. The first is by brute force; the second by using lag operators.

3.1. Solving by brute force. Consider equation (7) again:

$$m_t - e_t = y^* - \alpha(e_{t+1} - e_t). \tag{9}$$

The first step in solving forward is to solve for the current exchange rate in terms of next-period's exchange rate

$$e_t = \frac{m_t - y^*}{1+\alpha} + \frac{\alpha}{1+\alpha} e_{t+1}. \tag{10}$$

Equation (10) must hold for every t; hence it holds at $t+1$:

$$e_{t+1} = \frac{m_{t+1} - y^*}{1+\alpha} + \frac{\alpha}{1+\alpha} e_{t+2}. \tag{11}$$

Substitute equation (11) into equation (10):

$$e_t = \frac{m_t - y^*}{1+\alpha} + \frac{\alpha}{1+\alpha}\left(\frac{m_{t+1} - y^*}{1+\alpha} + \frac{\alpha}{1+\alpha} e_{t+2}\right). \tag{12}$$

Gather terms

$$e_t = \frac{1}{1+\alpha}\left(m_t + \frac{\alpha}{1+\alpha} m_{t+1}\right) - \frac{y^*}{1+\alpha}\left(1 + \frac{\alpha}{1+\alpha}\right) + \left(\frac{\alpha}{1+\alpha}\right)^2 e_{t+2}. \tag{13}$$

Do this again. By equation (10)

$$e_{t+2} = \frac{m_{t+2} - y^*}{1+\alpha} + \frac{\alpha}{1+\alpha} e_{t+3}. \tag{14}$$

Substitute equation (14) into equation (13):

$$e_t = \frac{1}{1+\alpha}\left(m_t + \frac{\alpha}{1+\alpha}m_{t+1}\right) - \frac{y^*}{1+\alpha}\left(1+\frac{\alpha}{1+\alpha}\right) + \left(\frac{\alpha}{1+\alpha}\right)^2\left(\frac{m_{t+2}-y^*}{1+\alpha} + \frac{\alpha}{1+\alpha}e_{t+3}\right). \quad (15)$$

Gather terms

$$e_t = \frac{1}{1+\alpha}\left[m_t + \frac{\alpha}{1+\alpha}m_{t+1} + \left(\frac{\alpha}{1+\alpha}\right)^2 m_{t+2}\right] - \frac{y^*}{1+\alpha}\left[1+\frac{\alpha}{1+\alpha}+\left(\frac{\alpha}{1+\alpha}\right)^2\right] + \left(\frac{\alpha}{1+\alpha}\right)^3 e_{t+3}. \quad (16)$$

We now notice a pattern; it we keep repeating the process we would get

$$e_t = \frac{1}{1+\alpha}\left[m_t + \frac{\alpha}{1+\alpha}m_{t+1} + ... + \left(\frac{\alpha}{1+\alpha}\right)^{T-1} m_{t+T-1}\right] - \frac{y^*}{1+\alpha}\left[1+\frac{\alpha}{1+\alpha}+ ... +\left(\frac{\alpha}{1+\alpha}\right)^{T-1}\right] + \left(\frac{\alpha}{1+\alpha}\right)^T e_{t+T}. \quad (17)$$

To see this, note that when we went out to the exchange rate e_{t+3}, we multiplied this exchange rate by a term raised to the power 3. Thus, when we go out to the exchange rate e_{t+T}, we multiply this exchange rate by a term raised to the power T. When we went out to the exchange rate e_{t+3}, we included money supplies and constant income out to the period $t+3-1=t+2$.Thus, when we go out to the exchange rate e_{t+T}, we include money supplies and constant income out to the period $t+T-1$.

Rewriting equation (17) using the summation notation yields

$$e_t = \frac{1}{1+\alpha}\sum_{s=0}^{T-1}\left(\frac{\alpha}{1+\alpha}\right)^s m_{t+s} - \frac{y^*}{1+\alpha}\sum_{s=0}^{T-1}\left(\frac{\alpha}{1+\alpha}\right)^s + \left(\frac{\alpha}{1+\alpha}\right)^T e_{t+T}. \quad (18)$$

Now suppose that we repeat this an infinite number of times. That is, we let T go to infinity.[3]

$$e_t = \frac{1}{1+\alpha}\sum_{s=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^s m_{t+s} - \frac{y^*}{1+\alpha}\sum_{s=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^s + \lim_{T\to\infty}\left(\frac{\alpha}{1+\alpha}\right)^T e_{t+T}. \tag{19}$$

Recall (or memorise!) the rule that

$$\sum_{s=0}^{\infty} x^s = \frac{1}{1-x} \text{ iff } |x| < 1. \tag{20}$$

Thus we have

$$\frac{y^*}{1+\alpha}\sum_{s=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^s = \frac{y^*}{1+\alpha}\frac{1}{1-\frac{\alpha}{1+\alpha}} = \frac{y^*}{1+\alpha-\alpha} = y^*. \tag{21}$$

Substituting equation (21) into equation (19) yields

$$e_t = \frac{1}{1+\alpha}\sum_{s=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^s m_{t+s} - y^* + \lim_{T\to\infty}\left(\frac{\alpha}{1+\alpha}\right)^T e_{t+T}. \tag{22}$$

The last term goes to zero unless the exchange rate goes to plus or minus infinity. Thus, our boundary condition is that the last term must equal zero. Thus our solution is

$$e_t = \frac{1}{1+\alpha}\sum_{s=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^s m_{t+s} - y^*. \tag{23}$$

This has intuitive appeal. The exchange rate depends on the current and all future values of the money supply. The importance of a future value of the money supply in determing the present exchange rate depends on how far out in the future it is. The further in the future, the less effect it has on the current exchange rate.

If we allow for bubbles then it can be shown that

$$e_t = \frac{1}{1+\alpha}\sum_{s=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^s m_{t+s} - y^* + c\left(\frac{1+\alpha}{\alpha}\right)^t, \tag{24}$$

[3] Technically, this requires that the money supply does not grow so fast that the limit of the sum involving the money supply fails to exist.

where c is any constant. Suppose that the money supply is constant then we can show that

$$e_t = m - y^* + c\left(\frac{1+\alpha}{\alpha}\right)^t \tag{25}$$

satisfies the equilibrium condition. Substitute this into $m - e_t = y^* - \alpha(e_{t+1} - e_t)$

$$\begin{aligned}
m - m + y^* - c\left(\frac{1+\alpha}{\alpha}\right)^t &= y^* - \alpha\left[m - y^* + c\left(\frac{1+\alpha}{\alpha}\right)^{t+1} - m + y^* - c\left(\frac{1+\alpha}{\alpha}\right)^t\right] \Leftrightarrow \\
-c\left(\frac{1+\alpha}{\alpha}\right)^t &= -\alpha\left[c\left(\frac{1+\alpha}{\alpha}\right)^{t+1} - c\left(\frac{1+\alpha}{\alpha}\right)^t\right] \Leftrightarrow \\
1 &= \alpha\left[\left(\frac{1+\alpha}{\alpha}\right) - 1\right] \text{ which is true}
\end{aligned}$$